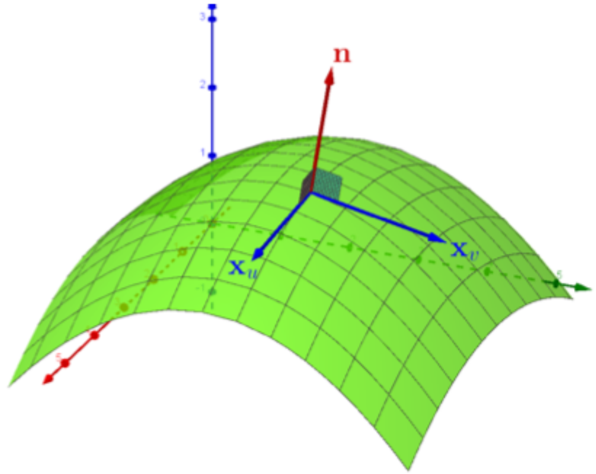
3.3 Second fundamental form and Gaussian curvature



Definition 3.3.1 (Unit normal vector). *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface. The* **unit normal vector** *to the surface is*

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

Definition 3.3.3 (Second fundamental form). *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface which has continuous second derivatives. The* **second fundamental form** *is the* 2×2 *matrix valued function*

$$II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & \langle \mathbf{x}_{uv}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & \langle \mathbf{x}_{vv}, \mathbf{n} \rangle \end{pmatrix} = - \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{n}_u \rangle & \langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{n}_u \rangle & \langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{pmatrix}.$$

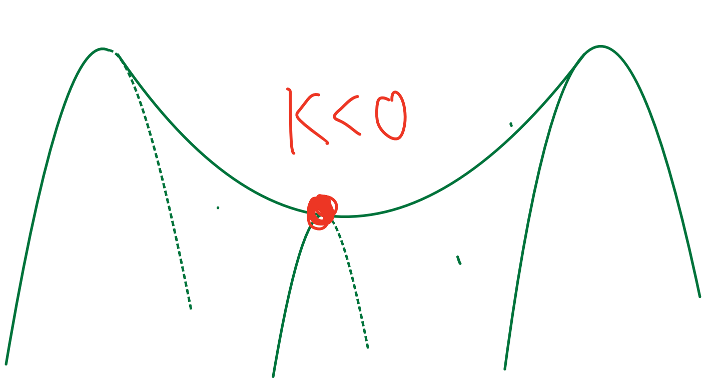
Definition 3.3.4 (Gaussian curvature). *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface which has continuous second derivatives. The* **Gaussian curvature** *of the surface is*

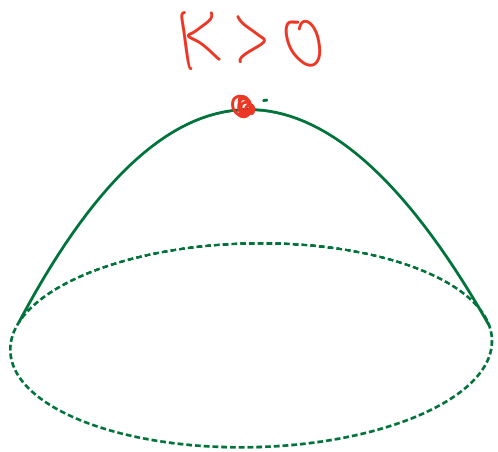
$$K = \frac{\det(II)}{\det(I)} = \frac{eg - f^2}{EG - F^2}$$

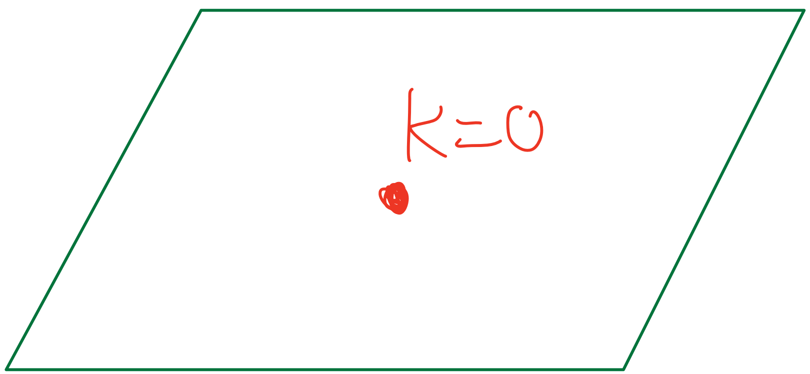
where I *is the first fundamental form and* II *is the second fundamental form of the surface.*

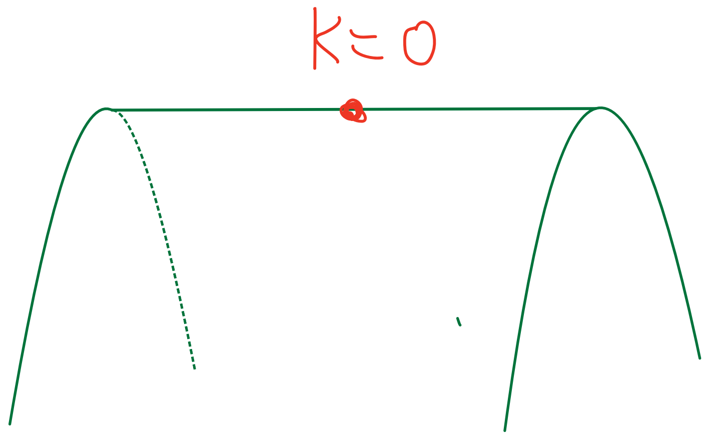
$$K = \frac{\langle r'', N \rangle}{\|r'\|^2}$$

$$K = \frac{\begin{vmatrix} \langle X_{uu}, n \rangle & \langle X_{uv}, n \rangle \\ \langle X_{vu}, n \rangle & \langle X_{vv}, n \rangle \end{vmatrix}}{\|X_u \times X_v\|^2}$$


K<0


K>0


K=0


K=0

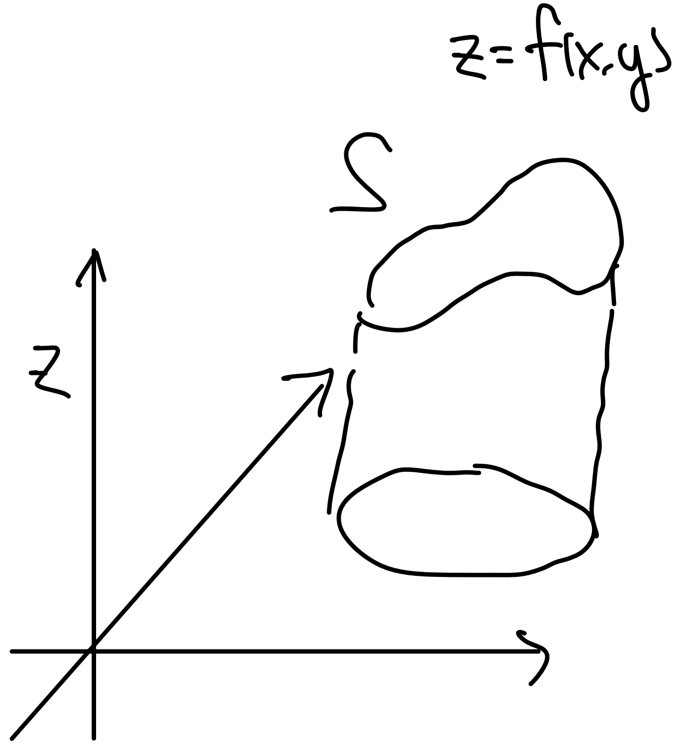
Proposition 3.3.6 (Curvature of graphs of functions).

1. *Let $f(x, y)$, $(x, y) \in D \subset \mathbb{R}^2$, be a function with continuous second derivatives. The Gaussian curvature of the graph of $z = f(x, y)$ in rectangular coordinates is*

$$K(x, y) = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

2. *Let $f(r, \theta)$, $(r, \theta) \in D \subset \mathbb{R}^+ \times (0, 2\pi)$, be a function with continuous second derivatives. The Gaussian curvature of the graph of $z = f(r, \theta)$ in cylindrical coordinates is*

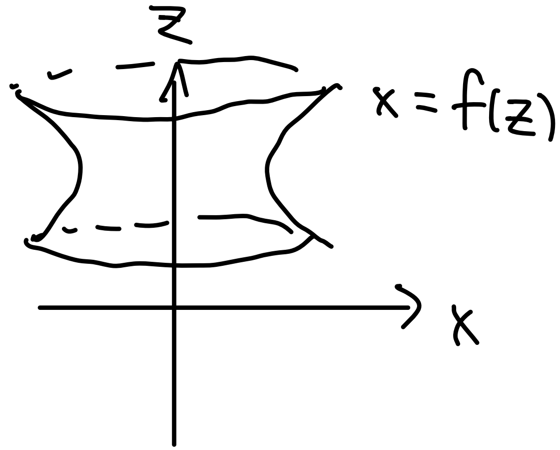
$$K(r, \theta) = \frac{r^2 f_{rr}(r f_r + f_{\theta\theta}) - (r f_{r\theta} - f_\theta)^2}{(r^2 + r^2 f_r^2 + f_\theta^2)^2}.$$



Proposition 3.3.7 (Gaussian curvature of surfaces of revolution).

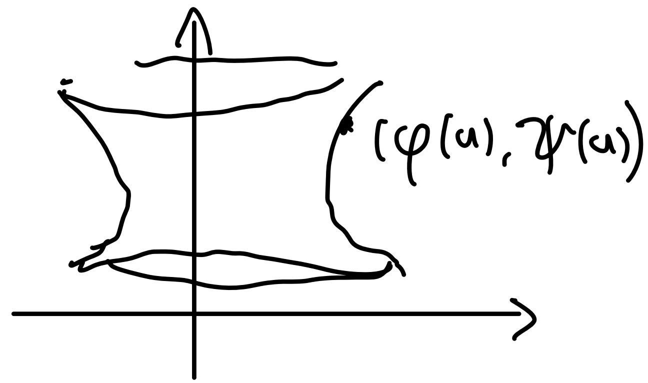
1. **By graph of function**: *Let $f(z)$, $z \in (a,b)$, be a function with continuous second derivative. The Gaussian curvature of the surface obtained by rotating the graph of $x = f(z)$ on the xz-plane about the z axis is*

$$K(z) = -\frac{f''}{f(1+f'^2)^2}.$$



2. **By parametrized curve**: *Let $(\varphi(u), \psi(u))$, $u \in (a,b)$, be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(u), \psi(u))$ on the xz-plane about the z axis is*

$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$



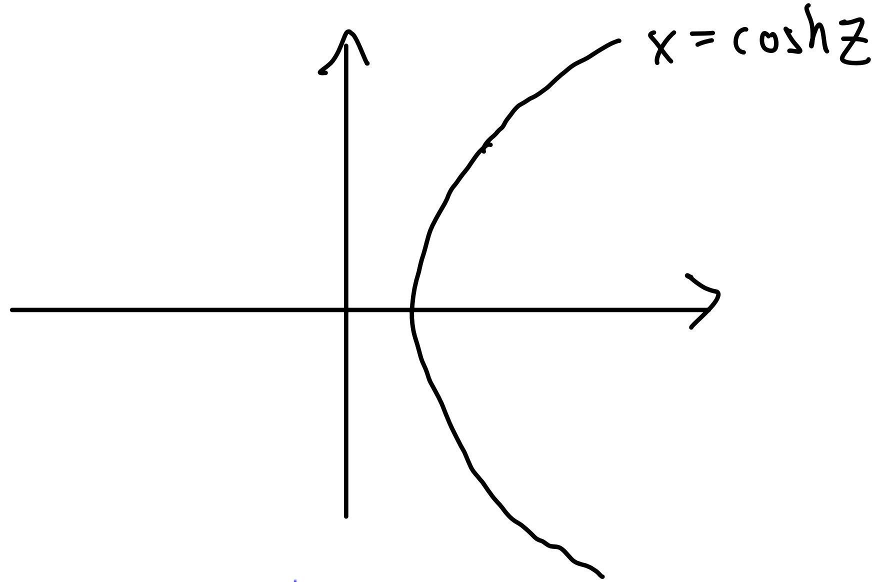
3. **By arc length parametrized curve**: *Let $(\varphi(s), \psi(s))$, $s \in (a,b)$, be an arc length parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(s), \psi(s))$ on the xz-plane about the z axis is*

$$K(s) = -\frac{\varphi''}{\varphi}.$$

$$\psi'^2 + \varphi'^2 = 1$$

Example 3.3.8 (Catenoid). *Consider the surface obtained by rotating the catenary $x = f(z) = \cosh z$ in the xz-plane about the z axis which is called* **catenoid**. *The Gaussian curvature of catenoid is*

$$K(z) = -\frac{f''}{f(1+f'^2)^2}.$$



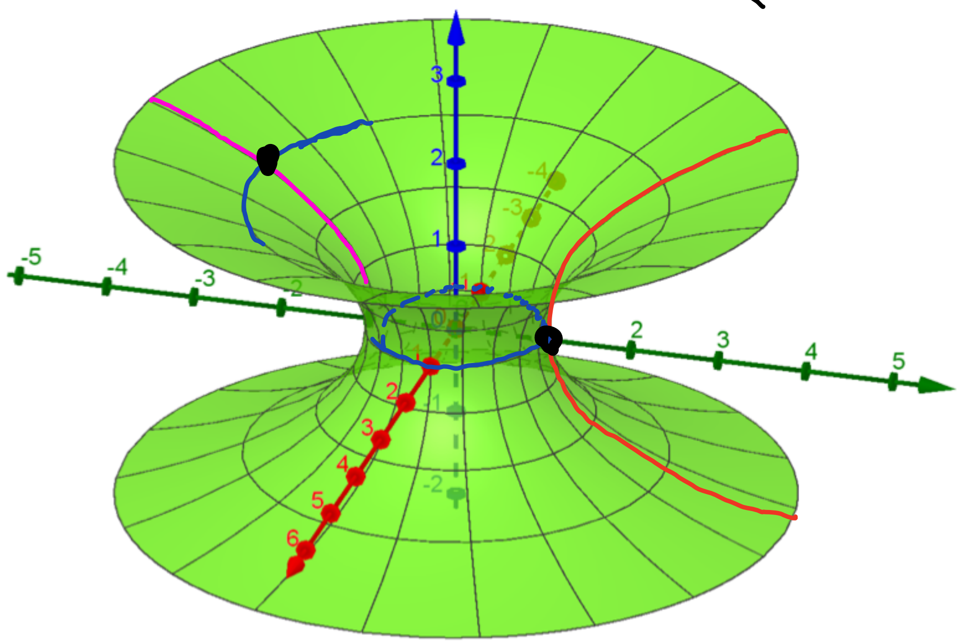
$f(z) = \cosh z$

$f'(z) = \sinh z$

$f''(z) = \cosh z$

$$K(z) = -\frac{\cosh z}{\cosh z\,(1+\sinh^2 z)^2}$$

$$= -\frac{1}{\cosh^4 z}$$

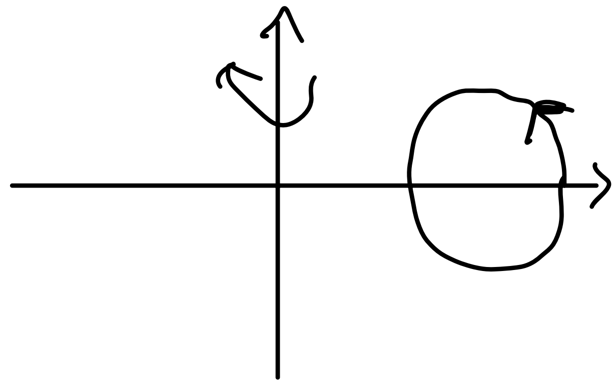


Example 3.3.9 (Torus). *Show that the Gaussian curvature of the torus obtained by rotating the arc length parametrized curve*

$$(x,z) = (\varphi(s), \psi(s)) = \left(R + r\sin\frac{s}{r}, r\cos\frac{s}{r}\right), \quad s \in (0, 2\pi)$$

about the z-axis is

$$K = \frac{\sin\dfrac{s}{r}}{r(R + r\sin\frac{s}{r})}.$$



$$(R + r\sin\phi, \ r\cos\phi)$$

$$K = -\frac{\varphi''}{\varphi}$$

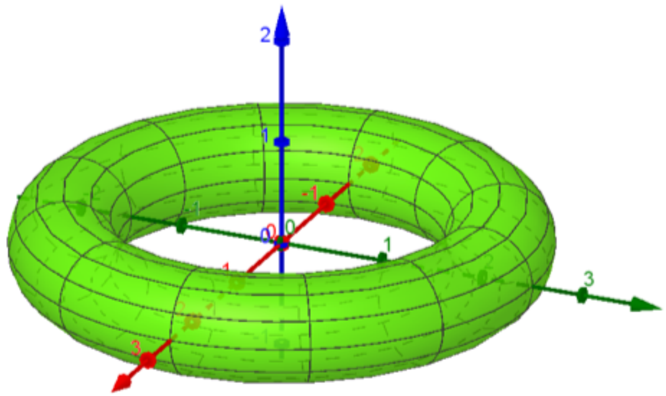
$$= -\frac{-\frac{1}{r}\sin\frac{s}{r}}{R + r\sin\frac{s}{r}}$$

$$= \frac{\sin\frac{s}{r}}{r(R + r\sin\frac{s}{r})}$$

$$\varphi(s) = R + r\sin\frac{s}{r}$$

$$\varphi'(s) = \cos\frac{s}{r}$$

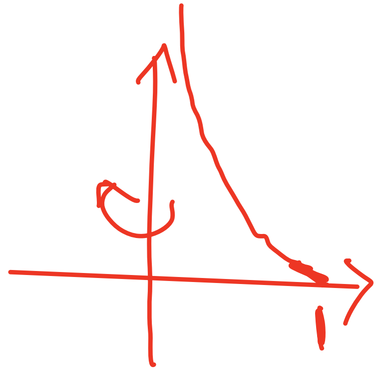
$$\varphi''(s) = -\frac{1}{r}\sin\frac{s}{r}$$



Example 3.3.10 (Pseudosphere). *Consider the surface obtained by rotating the tractrix (Example $\boxed{2.2.10}$)*

$$(x, z) = (\varphi(t), \psi(t)) = (\operatorname{sech} t, t - \tanh t), \quad t > 0$$

about the z-axis. This surface is called the **pseudosphere**. *Show that the pseudosphere has constant Gaussian curvature equal to -1.*



Reparametrized by arclength

$$(\varphi(s), \psi(s)) = \left(e^{-s}, \ln\left(e^{s} + \sqrt{e^{2s}-1}\right) - \sqrt{1-e^{-2s}}\right)$$

$$K = -\frac{\varphi''}{\varphi} = -\frac{e^{-s}}{e^{-s}} = -1$$

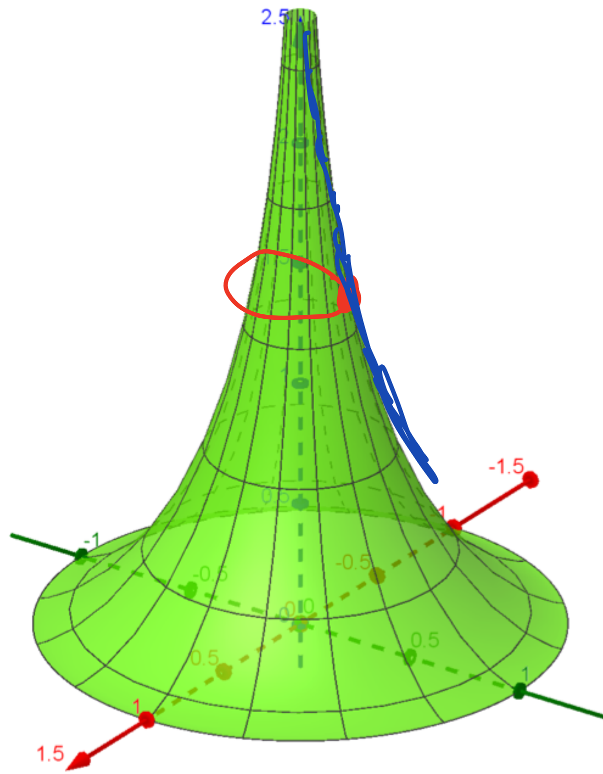
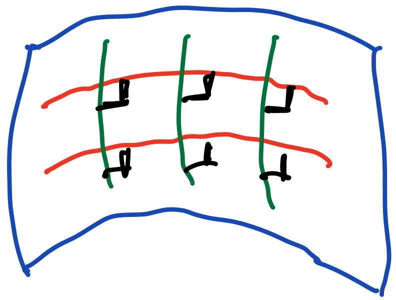


Figure 16: Pseudosphere

Theorem 3.3.11. *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface. Suppose* $\underbrace{F=0}$, *i.e., the first fundamental form of* $\mathbf{x}(u,v)$ *is*

$\langle X_u, X_v \rangle \equiv 0$

$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}.$$



Then the Gaussian curvature of $\mathbf{x}(u,v)$ *is*

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

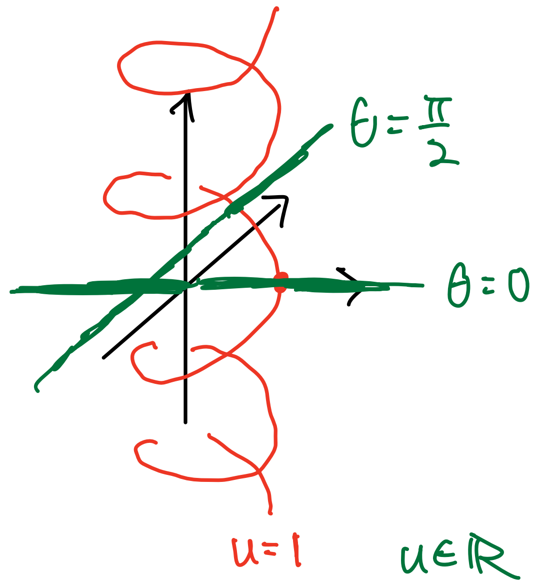
See note

Example 3.3.12 (Helicoid). *Show that the Gaussian curvature of the* **helicoid** *parametrized by*

$$\mathbf{x}(u,\theta) = (u\cos\theta, u\sin\theta, \theta), \;\; u, \theta \in \mathbb{R},$$

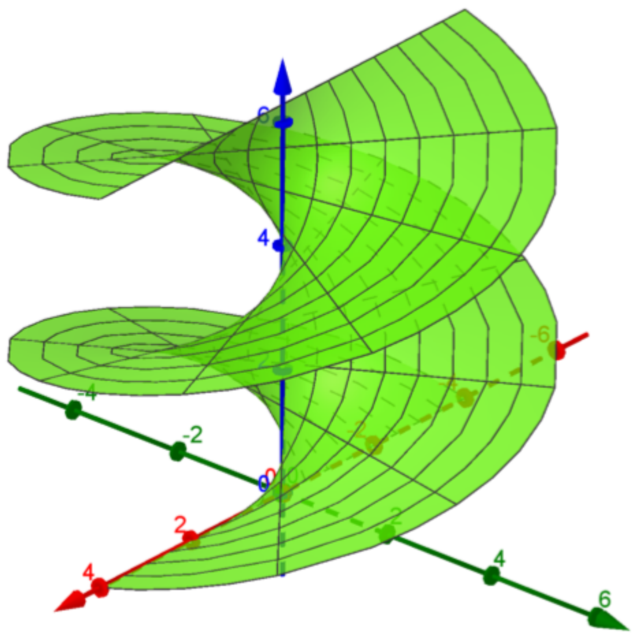
is

$$K = -\frac{1}{(1+u^2)^2}.$$



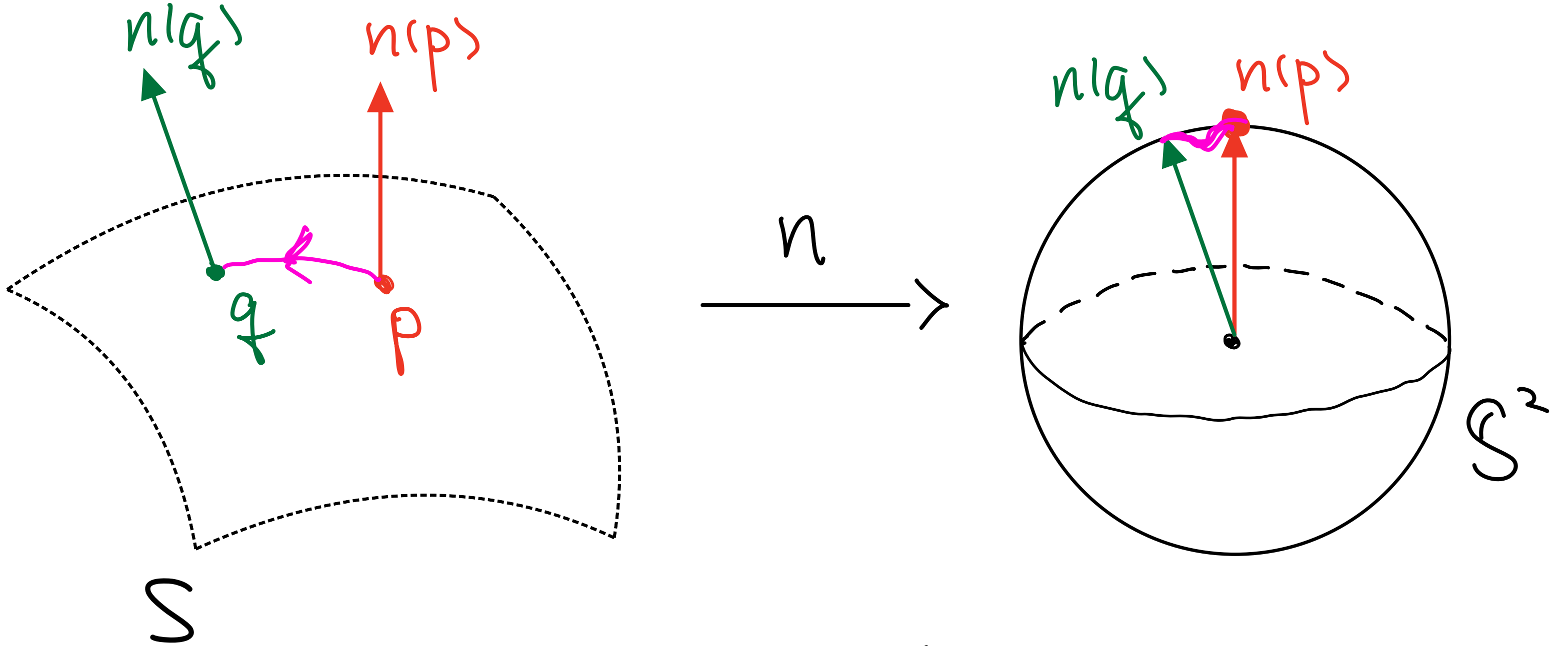
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1+u^2 \end{bmatrix} \qquad \begin{array}{l} E = 1 \\ G = 1+u^2 \end{array}$$

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$



3.4 Gauss map and its differential

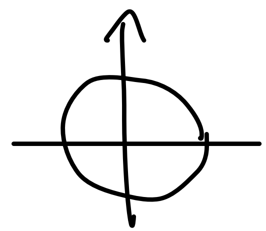
Definition 3.4.1 (Gauss map). *Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u, v)$. For each $p = \mathbf{x}(u, v)$, we associate the unit normal vector $\mathbf{n}(p)$ to p. This defines a map $\mathbf{n} : S \to S^2$ from the surface S to the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and is called the* **Gauss map** *of S.*

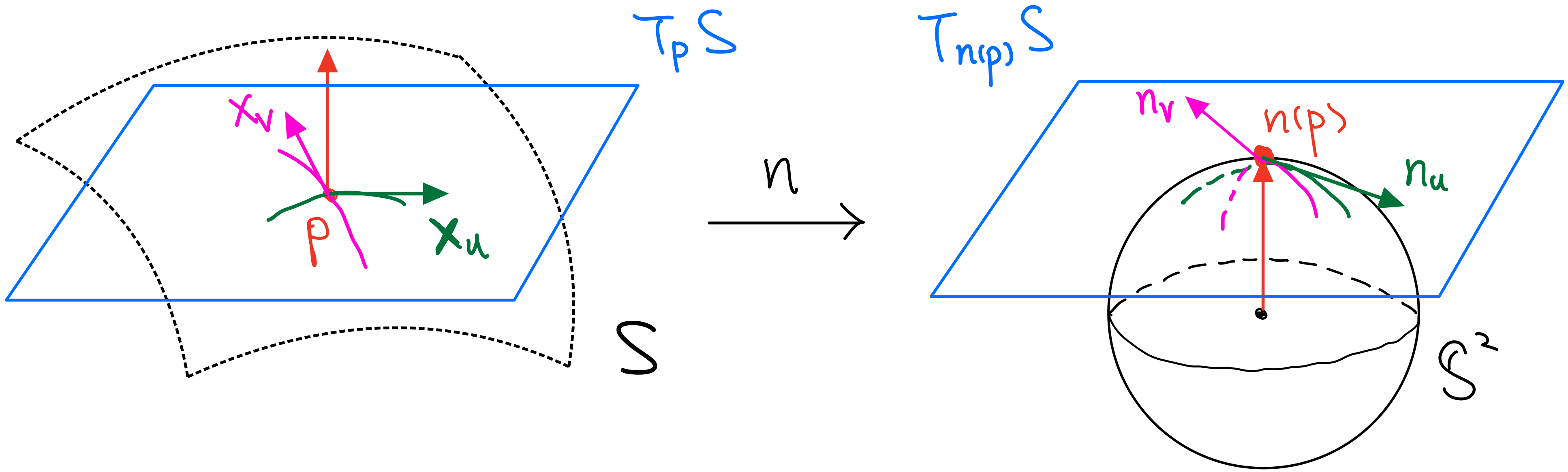


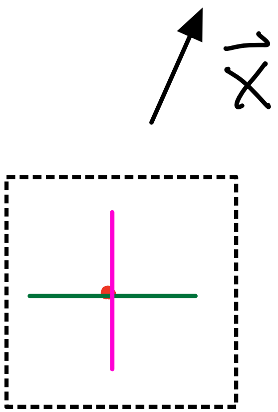
S^2 = sphere (2-dim)

S^1 = "sphere" (1-dim)

= circle in $\mathbb{R}^2$







$$T_pS = T_{n(p)}S^2 = \{\vec{v} \in \mathbb{R}^3 : \langle \vec{v}, n(p) \rangle = 0\}$$

$$n_u, n_v \in T_{n(p)}S^2 = T_pS$$

$$\because \|n\| \equiv 1 \Rightarrow \langle n, n \rangle \equiv 1 \Rightarrow \langle n, n \rangle_u = 0$$

$$\langle n_u, n \rangle + \langle n, n_u \rangle = 0 \Rightarrow \langle n_u, n \rangle = 0$$

Later: find a, b, c, d such that $\begin{cases} n_u = a x_u + b x_v \\ n_v = c x_u + d x_v \end{cases}$

Theorem 3.4.3. *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface and* $\mathbf{n}(u,v)$ *be the unit normal vector at* $\mathbf{x}(u,v)$. *Then*

$$\mathbf{n}_u \times \mathbf{n}_v = K\mathbf{x}_u \times \mathbf{x}_v$$

where K *is the Gaussian curvature of the surface.*

<u>Pf</u> $\quad n_u, n_v \perp X_u \times X_v \quad \Rightarrow \quad n_u \times n_v = c\, X_u \times X_v$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} \qquad \det \mathrm{I} = \|X_u \times X_v\|^2 = \langle X_u \times X_v, X_u \times X_v \rangle$$

$$\det(\mathrm{II}) = \begin{vmatrix} \langle X_u, n_u \rangle & \langle X_u, n_v \rangle \\ \langle X_v, n_u \rangle & \langle X_v, n_v \rangle \end{vmatrix}$$

<u>Formula</u>

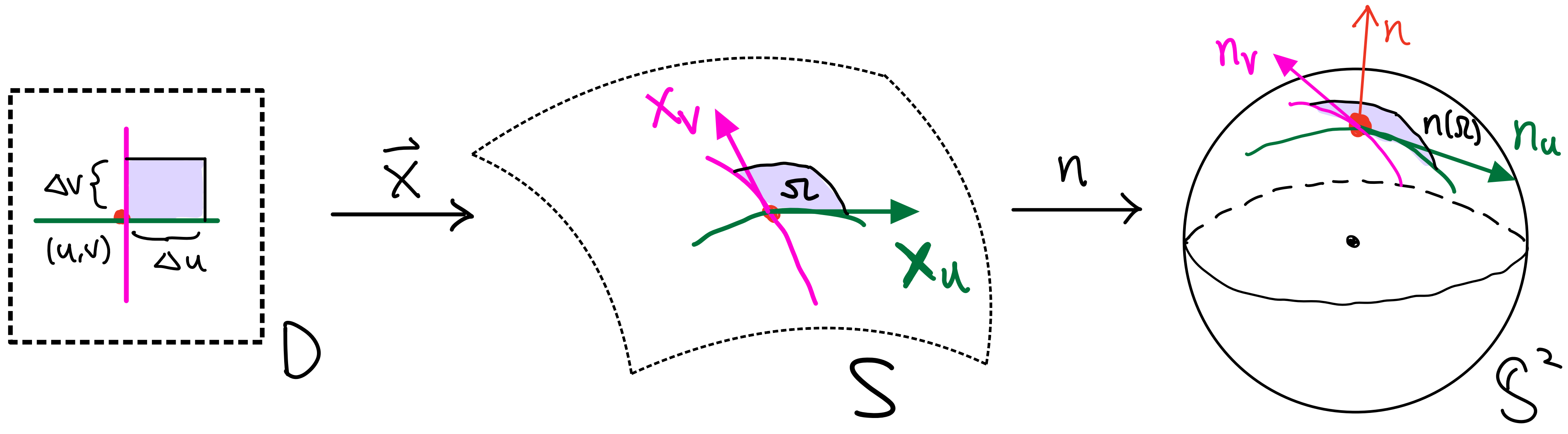
$$\begin{vmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{vmatrix} = \langle a \times b, c \times d \rangle$$

$$= \langle X_u \times X_v, n_u \times n_v \rangle$$

$$\mathrm{II} = \begin{bmatrix} -\langle X_u, n_u \rangle & -\langle X_u, n_v \rangle \\ -\langle X_v, n_u \rangle & -\langle X_v, n_v \rangle \end{bmatrix}$$

$$= c\, \langle X_u \times X_v, X_u \times X_v \rangle$$

$$= c \det \mathrm{I} \quad \Rightarrow \quad c = \frac{\det \mathrm{II}}{\det \mathrm{I}} = K$$



Define $\Omega = \{\mathbf{x}(s,t) : u < s < u + \Delta u, v < t < \Delta v\} \subset S$

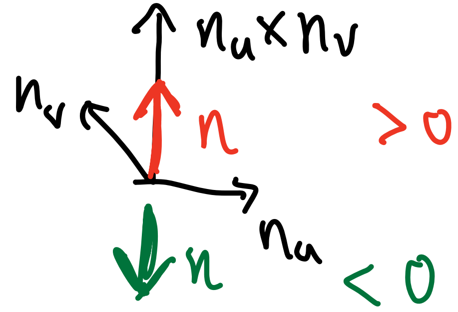
ΔA = Area of Ω

$$\Delta\sigma = \text{signed area of } n(\Omega) = \begin{cases} \text{Area of } n(\Omega) & \text{if } \underbrace{n_u, n_v, n \text{ satisfy right hand rule}} \\ -\text{Area of } n(\Omega) & \text{if } \underbrace{n_u, n_v, n \text{ satisfy left hand rule}}_{\text{orientation}} \end{cases}$$

$$\Delta A \approx \| \Delta u\, x_u \times \Delta v\, x_v \| = \| x_u \times x_v \| \Delta u \Delta v$$

$$|\Delta\sigma| \approx \| \Delta u\, n_u \times \Delta v\, n_v \| = \| n_u \times n_v \| \Delta u\, \Delta v = \| K\, x_u \times x_v \| \Delta u \Delta v = |K|\, \Delta A$$

Sign of $\Delta\sigma$ = sign of $\langle n_u \times n_v, n \rangle$



$$\langle n_u \times n_v, n \rangle = \left\langle K X_u \times X_v, \frac{X_u \times X_v}{\|X_u \times X_v\|} \right\rangle$$

$$= K \frac{\|X_u \times X_v\|^2}{\|X_u \times X_v\|} = K \|X_u \times X_v\| \begin{cases} > 0 & \text{if } K > 0 \\ < 0 & \text{if } K < 0 \end{cases}$$

$$\Rightarrow \quad \Delta\sigma \approx K \Delta A \qquad \frac{\Delta\sigma}{\Delta A} \approx K$$

Proposition 3.4.4. *Let S be a regular surface with parametrization $\mathbf{x}(u, v)$, $(u, v) \in D$. Let A and σ be the signed surface area function on S and S^2 respectively. Then we have*

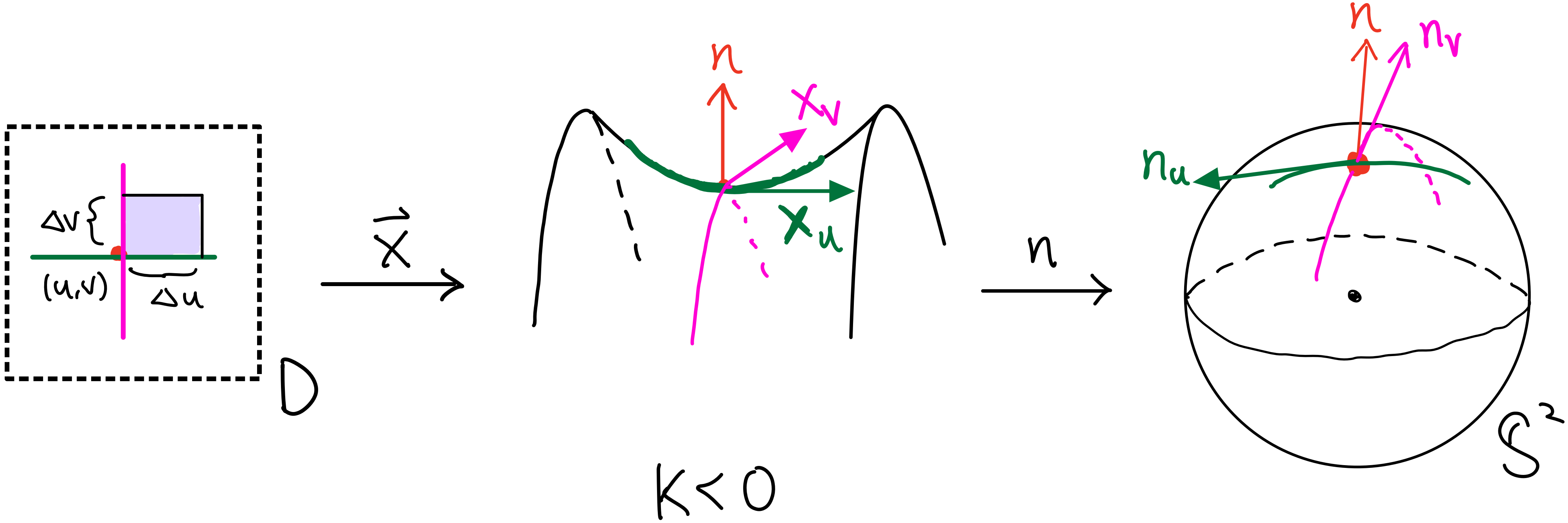
$$\frac{d\sigma}{dA} = K \quad \longleftarrow \quad \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \frac{\Delta\sigma}{\Delta A} = K$$

where K is the Gaussian curvature.

Proposition 3.4.4. *Let S be a regular surface with parametrization $\mathbf{x}(u, v)$, $(u, v) \in D$. Let A and σ be the signed surface area function on S and S^2 respectively. Then we have*

$$\frac{d\sigma}{dA} = K$$

where K is the Gaussian curvature.



n_u, n_v, n

left hand rule

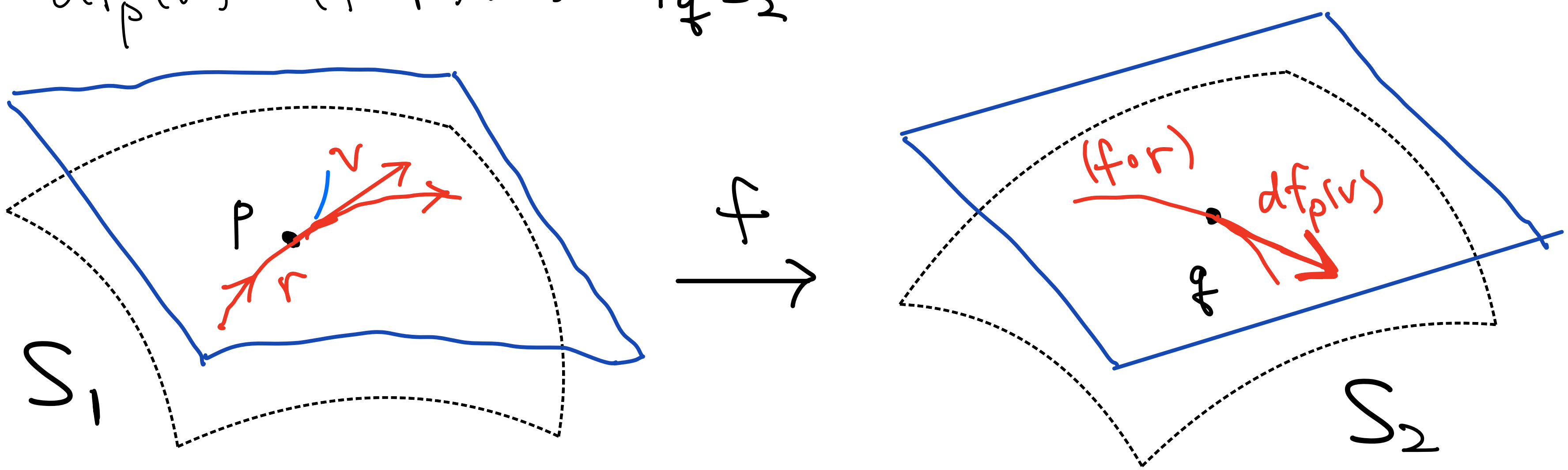
Differential

Suppose $f: S_1 \to S_2$ is a differentiable map between regular surfaces

Let $p \in S_1$, $q = f(p)$ and $v \in T_p S_1$.

Let r be a regular curve on S_1 with $r(0) = p$ and $r'(0) = v$

Define $df_p(v) = (f \circ r)'(0) \in T_q S_2$



<u>Fact</u> $df_p : T_p S_1 \to T_q S_2$ is linear

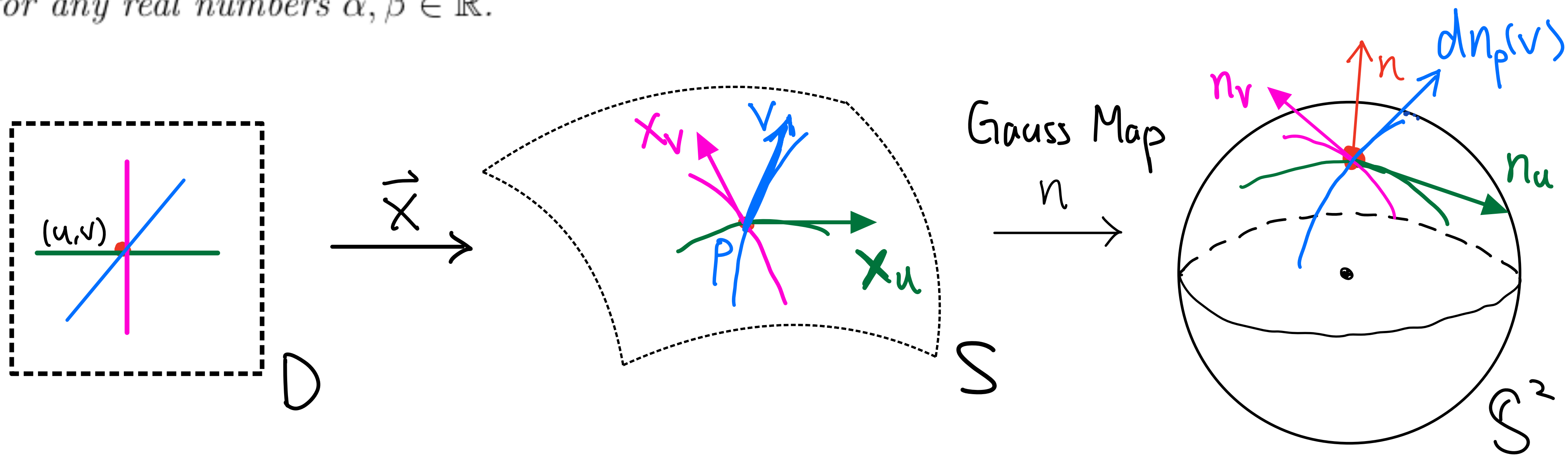
i.e $df_p(\alpha X_u + \beta X_v) = \alpha\, df_p(X_u) + \beta\, df_p(X_v)$

Differential of Gauss Map

Definition 3.4.5 (Differential of Gauss map). *Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u,v)$. For each $p \in S$, define $d\mathbf{n}_p : T_pS \to T_pS$ called the* **differential of Gauss map** *by*

$$d\mathbf{n}_p(\alpha\mathbf{x}_u + \beta\mathbf{x}_v) = \alpha\mathbf{n}_u + \beta\mathbf{n}_v$$

for any real numbers $\alpha, \beta \in \mathbb{R}$.



Proposition 3.4.2.

- $T_{\mathbf{n}}S^2 = T_pS$.
- $\mathbf{n}_u, \mathbf{n}_v \in T_pS$ $\Rightarrow \begin{cases} n_u = a x_u + b x_v \\ n_v = c x_u + d x_v \end{cases}$

Next : find a, b, c, d in terms of I, II

$$\begin{cases} n_u = a x_u + b x_v \cdots ① \\ n_v = c x_u + d x_v \cdots ② \end{cases}$$

$E = \langle x_u, x_u \rangle$

$e = -\langle x_u, n_u \rangle$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= -(\mathrm{II})(\mathrm{I}^{-1})$$

$$= -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$= -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$= -\frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix}$$

$① \Rightarrow \langle x_u, n_u \rangle = \langle x_u, a x_u + b x_v \rangle$

$= \langle x_u, a x_u \rangle + \langle x_u, b x_v \rangle = aE + bF$

$= a\langle x_u, x_u \rangle + b\langle x_u, x_v \rangle = aE + bF$

$\langle x_v, n_u \rangle = \langle x_v, a x_u + b x_v \rangle = aF + bG$

Similarly $② \Rightarrow \langle x_u, n_v \rangle = cE + dF$

$\langle x_v, n_v \rangle = cF + dG$

Rmk $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^2 \dfrac{\det \mathrm{II}}{\det \mathrm{I}} = K$

$$-\mathrm{II} = \begin{bmatrix} \langle x_u, n_u \rangle & \langle x_v, n_u \rangle \\ \langle x_u, n_v \rangle & \langle x_v, n_v \rangle \end{bmatrix}$$

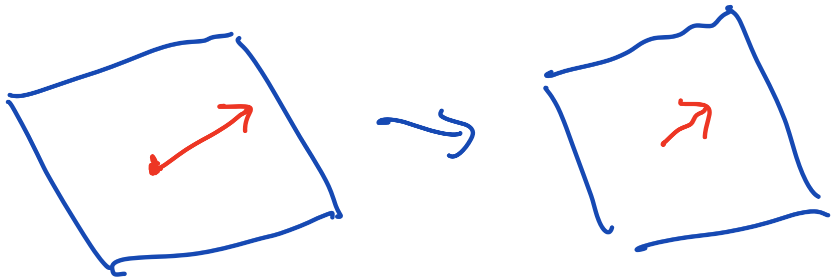
$$= \begin{bmatrix} aE+bF & aF+bG \\ cE+dF & cF+dG \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mathrm{I}$$

Proposition 3.4.9. *The matrix representation of $d\mathbf{n}_p$ with respect to basis $\mathbf{x}_u, \mathbf{x}_v$ is*

$$-(II)(I^{-1}) = -\frac{1}{EG - F^2}\begin{pmatrix} eG - fF & fE - eF \\ fG - gF & gE - fF \end{pmatrix}.$$

It means $dn_p(x_u) = a x_u + b x_v$

$dn_p(x_v) = c x_u + d x_v$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -(\mathrm{II})\, \mathrm{I}^{-1}$



$\underset{\substack{\uparrow \\ \text{eigenvalue}}}{}$ $Av = \lambda v$, $v \neq 0$
λ: eigenvalue, v: eigenvector

$dn_p : T_p S \longrightarrow T_{n(p)} S^2 = T_p S$

Theorem 3.4.7 (Self-adjointness of differential of Gauss map). *The differential of Gauss map $d\mathbf{n}_p : T_pS \to T_pS$ is self-adjoint. In other words, for any $\mathbf{u}, \mathbf{v} \in T_pS$, we have*

$$\langle d\mathbf{n}_p(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, d\mathbf{n}_p(\mathbf{v})\rangle.$$

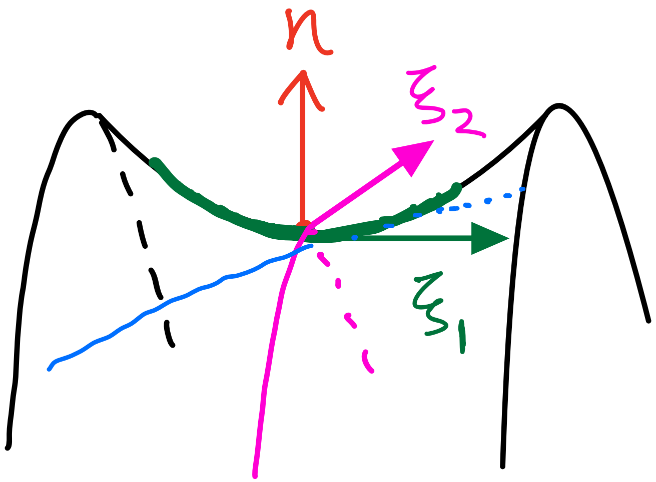
Definition 3.4.6 (Principal curvatures and principal directions). *Let S be a regular surface and $p \in S$. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ be two linearly independent eigenvectors of the differential $d\mathbf{n}_p : T_pS \to T_pS$ of Gauss map at p and κ_1, κ_2 be negative of the associated eigenvalues respectively. In other words,*

ξ xi

$$\begin{cases} d\mathbf{n}_p(\boldsymbol{\xi}_1) = -\kappa_1 \boldsymbol{\xi}_1 \\ d\mathbf{n}_p(\boldsymbol{\xi}_2) = -\kappa_2 \boldsymbol{\xi}_2 \end{cases}.$$

Then we say that κ_1, κ_2 are the **principal curvatures** *of S at p, and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are the corresponding* **principal directions**.

Theorem 3.4.8. *Let S be a regular surface in $\mathbb{R}^3$ and $p \in S$. Then there exists principal directions $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in T_pS$ which constitute an orthonormal basis for T_pS.*



$$dn_p(\xi_1) = \overbrace{-K_1}^{<0}\xi_1 \qquad K_1 > 0$$

$$dn_p(\xi_2) = \underbrace{-K_2}_{>0}\xi_2 \qquad K_2 < 0$$

$K > 0$ if the curve bends towards $\vec{n}$

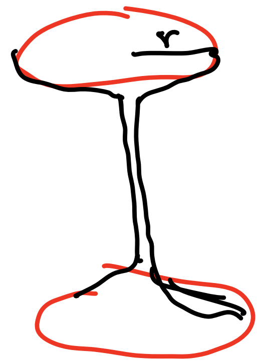
Theorem 3.4.10. *Let S be a regular surface and K be the Gaussian curvature of S. Then for any $p \in S$,*

$$K(p) = \det(d\mathbf{n}_p) = \kappa_1\kappa_2$$

Definition 3.4.11 (Mean curvature)**.** *Let S be a regular surface and $d\mathbf{n}_p$ be the differential of Gauss map at $p \in S$. The* **mean curvature** *of S at p is*

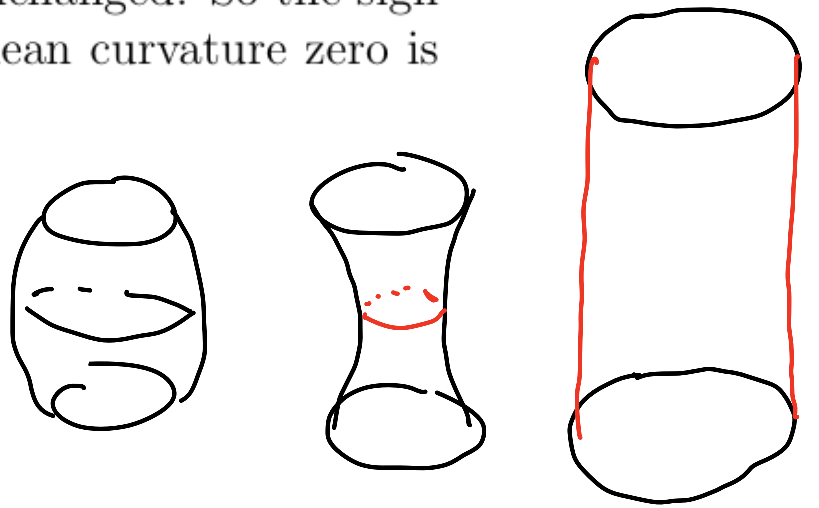
A ≈ $2\pi r^2$

$$H = -\frac{1}{2}\mathrm{tr}(d\mathbf{n}_p) = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\mathrm{tr}((II)(I^{-1})) = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right).$$



Note that if we reverse the direction of the unit vector $\mathbf{n}$, that is, reserving the order of the parameters u, v, there will be a change of sign of the mean curvature but the Gaussian curvature would remain unchanged. So the sign of mean curvature does not matter. A surface with mean curvature zero is called a minimal surface.

$H \equiv 0$



Definition 3.4.12 (Minimal surface). *Let S be a regular surface in $\mathbb{R}^3$ and H be the mean curvature of S. We say that S is a* **minimal surface** *if $H = 0$ at every point of S.*

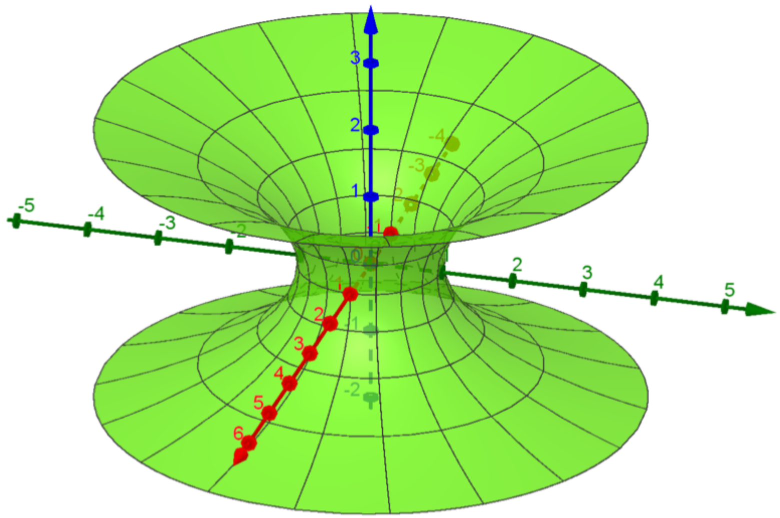
Theorem 3.4.13. *Let S be a minimal surface with parametrization $\mathbf{x} : D \to \mathbb{R}^3$ such that $\mathbf{x}$ can be extended continuously to the boundary. Then S has the minimum surface area among all surfaces with the same boundary of S.*

Example 3.4.14. *Show that the catenoid parametrized by*

$$\mathbf{x}(\theta, v) = (\cosh v \cos\theta, \cosh v \sin\theta, v), \ 1 < \theta < 2\pi, v \in \mathbb{R},$$

is a minimal surface.

$H \equiv 0$



Theorem 3.4.19. *Let S be a regular surface parametrized by $\mathbf{x}(u, v)$ and K be the Gaussian curvature of S.*

1.

$$K = \frac{\det(II)}{\det(I)}$$

where I and II are the first fundamental forms of S.

2.

$$\mathbf{n}_u \times \mathbf{n}_v = K\mathbf{x}_u \times \mathbf{x}_v$$

where $\mathbf{n}$ is the unit normal vector of S.

3.

$$K = \frac{d\sigma}{dA}$$

where A and σ are the signed area function on S and S^2 respectively.

4.

$$K = \kappa_1 \kappa_2$$

where κ_1, κ_2 are the principal curvatures associated with two orthogonal principal directions.